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A Global Optimization Problem in Series-Parallel Networks with Maximum Reliability

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In the reliability optimization of series-parallel networks (SPNs), the following problem occurs (see [3; 4]). Let $\mathcal{G}_n$ denote the set of SPNs with n components. An SPN is defined recursively as follows:

1. A single component, denoted by $\Box$, is an SPN.
2. If $N_1, \ldots, N_k$ $(k \geq 2)$ are SPNs, then also their arrangement in series, denoted by $S(N_1, \ldots, N_k)$, and their arrangement in parallel, denoted by $P(N_1, \ldots, N_k)$, are SPNs.

As an example, $S(P(\Box, \Box, \Box), P(S(\Box, \Box), \Box))$ is an SPN contained in $\mathcal{G}_6$. Its *nesting depth* (defined by the nesting depth of the brackets) is 3.

For $N \in \mathcal{G}_n$ let $f(N) = g(N) + h(N)$, where g, h are defined recursively by

$$g(\Box) = q,$$

$$g(S(N_1, \ldots, N_k)) = 1 - \prod_{i=1}^{k}(1 - g(N_i)),$$

$$g(P(N_1, \ldots, N_k)) = \prod_{i=1}^{k} g(N_i),$$

and

$$h(\Box) = s,$$

$$h(S(N_1, \ldots, N_k)) = \prod_{i=1}^{k} h(N_i),$$

$$h(P(N_1, \ldots, N_k)) = 1 - \prod_{i=1}^{k}(1 - h(N_i)),$$

with fixed numbers q, s $(q > 0,\ s > 0,\ q + s < 1)$. We consider the optimization problem

$$f(N) \to \min, \qquad N \in \mathcal{G}_n, \tag{1}$$

for given n, q, s.

In the special case of SPNs with nesting depth two, (1) reduces to the problem

$$\prod_{i=1}^{n}(1-(1-q)^{x_i}) - \prod_{i=1}^{n}(1-s^{x_i}) \to \min, \tag{2}$$

$$\text{s. t.} \quad \sum_{i=1}^{n} x_i = n, \tag{3}$$

$$x_i \in \{0, 1, \ldots\} \qquad (i = 1, \ldots, n). \tag{4}$$

(Note that the number of distinct function values of (2) is bounded by the number of partitions of the integer n.)

In [1], a *continuous relaxation* of (2) – (4) is investigated, (4) being replaced by the constraint

$$x_i \in \{0\} \cup [1, \infty[. \tag{5}$$

It is shown that in a solution vector $(x_1^*, \ldots, x_n^*)$ of the relaxed problem, only two different non-vanishing component values (depending on q, s and n) can occur. This allows a fairly efficient solution of the relaxed and the original "nesting depth two" problem.

Page and Perry [4] compute solutions for the *general* problem (no restriction on the nesting depth) up to about $n = 9$ components. In [2], the general problem is attacked by Discrete Optimization methods (improved enumeration, Simulated Annealing, Genetic Algorithms). Improved enumeration, based on suitable equivalence classes of rooted tree representations of feasible solutions, is fast up to about $n = 15$ components. However, no fast exact optimization algorithm for *large* n seems to be known up to now.

Answers to the following questions would be of particular interest:

1. Is there a continuous relaxation, treatable by Global Optimization methods, also for the general problem (1)? If yes, is it possible to derive general properties of the solution?
2. Is it possible to get more information on the solution of the problem (2), (3), (5)? E. g., what can be said about the number m of non-vanishing components x_i^* in a globally optimal solution?
3. What is the theoretical complexity of the problems (1) resp. (2) – (4) ?

References

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